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M theory, orientifolds and G-flux

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Abstract: We study the properties of M and F theory compactifications to three and four dimensions with background fluxes. We provide a simple construction of supersymmetric vacua, including some with orientifold descriptions. These vacua, which have warp factors, typically have fewer moduli than conventional Calabi-Yau compactifications. The mechanism for anomaly cancellation in the orientifold models involves background RR and NS fluxes. We consider in detail an orientifold of $K3 \times T^2$ with background fluxes. After a combination of T and S-dualities, this type IIB orientifold is mapped to a compactification of the SO(32) heterotic string on a non-Kähler space with torsion.

Keywords: M-Theory, F-Theory, Superstring Vacua, Supergravity Models.

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Contents

1. Introduction

The realization that certain string compactifications can be described in multiple ways has provided a significant improvement in our understanding of non-perturbative string dynamics. Among the more interesting compactifications are those with $N = 1$ and $N = 2$ supersymmetry in four dimensions. An intriguing class of such four dimensional vacua are described by compactifying F theory on a Calabi-Yau four-fold $\mathcal{M}$ [1]. These vacua are naturally related to compactifications of type IIA string theory and M theory to two and three dimensions, respectively. The strong coupling limit of type IIA compactified on $\mathcal{M}$ is well described by the three dimensional theory obtained by compactifying M theory on the same four-fold. If the four-fold admits an elliptic fibration with base $\mathcal{B}$, then we can consider a particular degeneration of this M theory compactification in which the area of the elliptic fiber shrinks to zero. In this limit, M theory on the four-fold goes over to type IIB compactified on the base $\mathcal{B}$ with a varying coupling constant. The coupling constant is τ of the elliptic fiber. This four-dimensional type IIB vacuum is known as an F theory compactification.

Among the novel features of these vacua is the need to cancel a tadpole anomaly. For this class of compactifications, the anomaly is given by $\chi/24$, where χ is the Euler characteristic of the four-fold. If $\chi/24$ is integral, then the anomaly can be cancelled by placing a sufficient number of spacetime filling branes on points of the compactification manifold [2]. For type IIA, the required branes are strings, while in M and F theory, membranes and $D3$-branes are required, respectively.

However, there is at least one other way of cancelling the anomaly in type IIA or M theory, which is by introducing a background flux for the four-form field strength G [3]. The G-flux contributes to the membrane tadpole in M theory through the Chern-Simons interaction,

$$\int C \wedge G \wedge G \,. \tag{1.1}$$

For cases where $\chi/24$ is not integral, G-flux is actually required to obtain a consistent vacuum. In general, the anomaly can be cancelled by a combination of background flux and branes. With n background branes, the tadpole cancellation condition

$$\frac{\chi}{24} = \frac{1}{8\pi^2} \int G \wedge G + n \,, \tag{1.2}$$

must be satisfied for type IIA or M theory [4].

Compactifications with G-flux have a number of interesting features, but have received little attention. The goal of this paper is to explore some of the properties of these vacua. In the following section, we review the results of [3] where conditions on supersymmetric vacua with G-flux were derived from eleven-dimensional supergravity. These conditions are quite difficult to satisfy. As a consequence, the presence of G-flux typically freezes some of the geometric moduli of the four-fold in a way that we describe.

Some of these M theory vacua can be lifted to four-dimensional $N = 1$ F theory vacua. The corresponding F theory vacua have background fluxes of two kinds. The first kind involves non-zero NS and RR three-form field strengths, denoted H and H' respectively. The three-form fluxes contribute to the $D3$-brane tadpole through the type IIB supergravity interaction,

$$\int D^+ \wedge H \wedge H' \,, \tag{1.3}$$

where D^+ is the four-form gauge field. In the second kind of background, some of the seven-brane gauge fields have non-zero instanton number. In F theory, this possibility has been discussed in [5]. These instantons contribute to the $D3$-brane tadpole through the seven-brane world-volume coupling,

$$\int D^+ \wedge F \wedge F \,, \tag{1.4}$$

where F is the field strength for the seven-brane gauge-field.

F theory is a useful description of these compactifications only when the base $\mathcal{B}$ is large compared to the string scale. However, for special choices of four-fold $\mathcal{M}$, F theory can be related to a type IIB orientifold which is a complete perturbative string theory [6]. In turn, some of these IIB orientifolds can be related to type I compactifications via T-duality. Largely for their simplicity, orientifolds of tori are most commonly studied. In conventional models, tadpole cancellation is achieved by adding branes: either $D9$ and $D5$-branes or $D7$ and $D3$-branes depending on the choice of orientifold action. In four dimensions, the possibility of using the type IIB interactions (1.3) and (1.4) to cancel the $D3$-brane tadpole suggests that novel orientifolds should exist with backgrounds involving H and H'-fluxes and gauge-field instantons. The possible types of orientifold are then classified by the choice of C-flux and G-flux in M theory on $\mathcal{M}$.

In section 3, we present examples of vacua with G-flux, including a simple orbifold construction and an example where G-flux is required. In the final section, we consider an example of a type IIB orientifold with constant background H and H'-fluxes. Depending on the choice of background flux, the model has either $N = 1$ or $N = 2$ spacetime supersymmetry. The compactification space $\mathcal{B}$ is conformal to $K3 \times T^2$. This orientifold is related to F theory on $K3 \times K3$, some aspects of which have been discussed in [7].

However, our interest is largely with a heterotic dual of this orientifold. By T-dualizing along the T^2, we map the IIB orientifold to a type I compactification on a new space $\mathcal{B}'$ with non-zero H'-torsion. A further S-duality turns the type I vacuum into a perturbative SO(32) heterotic vacuum with non-zero H-flux. The non-Kähler space $\mathcal{B}'$ is no longer conformally Calabi-Yau. This is a concrete example, possibly the first, of a four-dimensional string compactification with torsion.

There are a number of promising directions to explore. For example, associated to each of the type I/heterotic supergravity solutions is a world-sheet conformal field theory. Finding ways of constructing and analyzing these conformal field theories is a potentially rewarding enterprise. There should be analogues in this more general class of compactifications of phenomena associated with Calabi-Yau compactifications, such as mirror symmetry and its $(0, 2)$ cousin [8, 9]. There are also intriguing connections between these solutions and the work of [10, 11, 12]. During the completion of this project, an interesting paper appeared [13] with some overlap with section 2.

2. Supersymmetry and G-flux

2.1 C-field instantons

We begin by recalling the results of [3] for M theory compactified on an eight-dimensional Calabi-Yau manifold $\mathcal{M}$. Let M_{pl} denote the eleven-dimensional Planck scale. At leading order in a momentum expansion, the M theory effective action is given by eleven-dimensional supergravity. A product metric on $\mathbb{R}^3 \times \mathcal{M}$ is a solution to the supergravity equations of motion when the metric for the internal space $\mathcal{M}$ is Ricci flat. Let us parametrize $\mathbb{R}^3$ by coordinates x^μ where $\mu = 0, 1, 2$ and the internal space $\mathcal{M}$ by complex coordinates y^a where $a = 1, \ldots, 4$.

At next order in the derivative expansion, there are terms with eight derivatives which are therefore suppressed by six additional powers of M_{pl}. Among these terms is an interaction of the form [14, 15],

$$-\int C \wedge X_8(R)\,, \tag{2.1}$$

where X_8 is an eight-form constructed from curvature tensors. This term induces a tadpole for the C-field. A way to cancel the tadpole is to turn on a non-trivial G-flux. The metric is then modified from a simple product and takes the form:

$$ds^2 = e^{-\phi(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{\frac{1}{2}\phi(y)}g_{a\bar{b}}dy^a dy^{\bar{b}}\,. \tag{2.2}$$

The metric g for $\mathcal{M}$ is Kähler and Ricci flat. Let us call the warped internal space $\widehat{\mathcal{M}}$. The space $\widehat{\mathcal{M}}$ is therefore conformal to a Calabi-Yau manifold.

There is also a non-vanishing four-form field strength G on $\mathcal{M}$ which satisfies the conditions:

$$G_{abcd} = G_{abc\bar{d}} = 0\,, \qquad g^{c\bar{d}}G_{a\bar{b}c\bar{d}} = 0\,. \tag{2.3}$$

The only other non-vanishing component of G is given in terms of the warp factor,

$$G_{\mu\nu\rho a} = \epsilon_{\mu\nu\rho}\,\partial_a e^{-\frac{3}{2}\phi}\,. \tag{2.4}$$

Lastly, the warp factor satisfies the equation:

$$\Delta(e^{\frac{3\phi}{2}}) = *\left\{4\pi^2 X_8 - \frac{1}{2}G\wedge G - 4\pi^2\sum_{i=1}^{n}\delta^8(y-y_i)\right\}\,. \tag{2.5}$$

The laplacean and the Hodge star operator in (2.5) are defined with respect to g. We have included the possibility of n membranes located at the points y_i on $\widehat{\mathcal{M}}$. The combination of G-flux and membranes must satisfy (1.2).

The first condition in (2.3) tells us that G is a $(2,2)$ form on $\mathcal{M}$. Dirac quantization requires that the cohomology class $[G/\pi]$ be an element of $H^{(2,2)}(\mathcal{M},\mathbb{Z})$. If $\chi/24 \in \mathbb{Z}$ then $[G/2\pi]$ is an integer cohomology class [16]. The second condition in (2.3) is the analogue of the following condition for a two-form field strength F:

$$g^{a\bar{b}}F_{a\bar{b}} = 0\,. \tag{2.6}$$

In the two-form case, (2.6) implies that F is anti-self-dual because,

$$(*F)_{a\bar{b}} = \epsilon_{ac\bar{b}\bar{d}}F^{c\bar{d}} = (g_{a\bar{b}}g_{c\bar{d}} - g_{a\bar{d}}g_{c\bar{b}})F^{c\bar{d}} = -F_{a\bar{b}}\,. \tag{2.7}$$

In the four-form case, we can again express the epsilon tensor in the following way:

$$\epsilon_{abcd\bar{p}\bar{q}\bar{r}\bar{s}} = g_{a\bar{p}} g_{b\bar{q}} g_{c\bar{r}} g_{d\bar{s}} \pm \text{ permutations}.$$

In much the same way as the two-form case (2.7), the conditions (2.3) imply that,

$$G = *G, \tag{2.8}$$

where the Hodge star acts on the internal eight manifold with metric g. This M theory background therefore involves an abelian 'instanton' of the C-field. Lastly, let us rephrase the second condition (2.3) in terms of the Kähler form of $\mathcal{M}$,

$$J \equiv i g_{a\bar{b}} dz^a \wedge dz^{\bar{b}}. \tag{2.9}$$

In terms of J, the second condition states that the self-dual G-field is primitive:

$$J \wedge G = 0. \tag{2.10}$$

This condition actually means that G is a singlet of the sl_2-algebra generated by J, its adjoint $J^\dagger$ and their commutator $[J, J^\dagger]$. See, for example, [17].

2.2 Compactifications with extended supersymmetry

For compactifications on $\mathbb{R}^3 \times \mathcal{M}$, we decompose a 32 component Majorana-Weyl spinor under $\mathrm{SO}(2,1) \times \mathrm{SO}(8)$ in the following way,

$$\mathbf{32} = (\mathbf{2}, \mathbf{8_s}) \oplus (\mathbf{2}, \mathbf{8_c}). \tag{2.11}$$

For spaces $\mathcal{M}$ with holonomy SU(4) and not a proper subgroup, we can further decompose the SO(8) representations under SU(4):

$$\mathbf{8_s} = \mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1}, \mathbf{8_c} = \mathbf{4} \oplus \bar{\mathbf{4}}. \tag{2.12}$$

The two singlets in $\mathbf{8_s}$ give the four real unbroken supersymmetries needed for a model with $N = 2$ supersymmetry.

If the holonomy of $\mathcal{M}$ is a proper subgroup of SU(4), the theory may have extended supersymmetry. See [18] for a discussion of the possible holonomies of an eight manifold. We will discuss two examples which appear in later discussion. The first is compactification on a hyperKähler manifold with holonomy Sp(2). On decomposing the $\mathbf{8_s}$ and $\mathbf{8_c}$ representations, we find

$$\mathbf{8_s} = \mathbf{5} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}, \mathbf{8_c} = \mathbf{4} \oplus \mathbf{4}. \tag{2.13}$$

Therefore, this compactification can have $N = 3$ supersymmetry. Since the space is hyperKähler, there is a $\mathbb{P}^1$ of choices of complex structure. To see this, note that we

JHEP08(1999)023

can construct a complex structure tensor ${J^i}_j$ from any complex covariantly constant spinor η in the usual way:

$$ {J^i}_j = i\eta^\dagger {\gamma^i}_j \eta \,. \tag{2.14} $$

The indices $i, j = 1, \ldots, 8$ and γ^i are gamma matrices for $\mathcal{M}$. For each of the $\mathbb{P}^1$ of choices for η, where η has norm one, there is a corresponding complex structure tensor.

The second example has holonomy $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. In this case,

$$ \begin{aligned} \mathbf{8_s} &= (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \,, \\ \mathbf{8_c} &= (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}) \,. \end{aligned} \tag{2.15} $$

This compactification has at most $N = 4$ supersymmetry. There is a $\mathbb{P}^1 \times \mathbb{P}^1$ of complex structures. Whether there is extended supersymmetry actually depends on the choice of G-flux. To preserve more than $N = 2$ supersymmetry, the G-flux must be a primitive $(2, 2)$ class with respect to more than a single complex structure. In section 3, we will meet examples of G-flux which do not preserve all the supersymmetries of the compactification manifold.

2.3 Kaluza-Klein reduction with a warp factor

In the presence of background fluxes and a warp factor, the counting of light degrees of freedom is typically quite difficult because the equations obeyed by the metric and C-field fluctuations are coupled. The metric takes the form,

$$ \begin{pmatrix} e^{-\phi}\eta_{\mu\nu} & 0 \\ 0 & \widehat{g} \end{pmatrix}, \tag{2.16} $$

where $\widehat{g} = e^{\frac{1}{2}\phi} g$. Let us begin by considering purely metric deformations. Since the metric g is Calabi-Yau, infinitesimal deformations of g are classified in the usual way by elements of $H^{3,1}(\mathcal{M})$ and $H^{1,1}(\mathcal{M})$. However, those complex structure deformations that do not keep G a $(2, 2)$ class become massive. Likewise, deformations of the Kähler structure that do not keep G primitive become massive. In this way, we generically lose a large number of geometric moduli.

Let us take the generic case where $N = 2$ spacetime supersymmetry is preserved. The moduli that we wish to count appear in two kinds of multiplet: the first is the dimensional reduction of a four-dimensional $N = 1$ chiral multiplet. The second is the reduction of an $N = 1$ vector multiplet. In three dimensions, the vector multiplet can be dualized to a chiral multiplet containing a dual scalar. Each surviving deformation in $H^{3,1}(\mathcal{M})$ gives rise to a chiral multiplet, while each deformation in $H^{1,1}(\mathcal{M})$ gives rise to the scalar field of a vector multiplet. The vector field itself comes from a C-field zero-mode.

Let us consider the effect of just a warp factor on the C-field equations. Without G-flux, the C-field obeys the free-field equation,

$$d\,\hat{*}\,dC = 0\,. \tag{2.17}$$

By using the gauge invariance $C \to C + d\Lambda$, we can demand that C satisfy

$$d\,\hat{*}\,C = 0\,. \tag{2.18}$$

Combined with the field equation, this gives the usual condition,

$$\widehat{\Delta} C = 0\,. \tag{2.19}$$

Decomposing (2.19) into spacetime and internal components gives,

$$\left\{ \partial^\mu \partial_\mu + e^{-\phi}\widehat{\Delta}_g - \frac{3}{2} e^{-\phi}\, \widehat{g}^{ab}(\partial_a \phi)\partial_b \right\} C = 0\,. \tag{2.20}$$

The last two terms in (2.20) will look like mass terms from the perspective of the three-dimensional observer. The second term is conventional and leads to the usual harmonicity condition on the internal components of C. However, the third term is a new consequence of the warp factor.

In the presence of non-trivial background G-flux, equation (2.17) for a fluctuation δC becomes:

$$d\,\hat{*}\,d\,\delta C = -G \wedge d\,\delta C\,. \tag{2.21}$$

We have set all metric fluctuations to zero in (2.21). We decompose δC into a product of spacetime and internal fields,

$$\delta C = \psi(x) C^{(3)}(y) + A_\mu(x) C^{(2)}(y)\,, \tag{2.22}$$

where we only consider spacetime multiplets with vector or scalar fields, and $C^{(n)}$ is an n-form on the eight manifold. We can dispense immediately with the counting of vector fields $A_\mu(x)$ since each zero-mode of $C^{(2)}$ pairs with a Kähler class deformation to give a vector multiplet. We only need to count the number of massless modes from $H^{1,1}(\mathcal{M})$ to count the number of vector multiplets.

The final source of moduli are the analogues of Wilson lines for the C-field. In the absence of G-flux, any element of $H^{2,1}(\mathcal{M})$ gives rise to a zero-mode for $C^{(3)}$ and therefore a chiral multiplet.[1] In the presence of G-flux, the conditions on $C^{(3)}$ are modified. We can expand the left hand side of (2.21) as follows,

$$d\,\hat{*}\,d\,\delta C = d * d\psi \wedge *C^{(3)} + *d\psi \wedge d * C^{(3)} + *\psi \wedge d(e^{-3\phi/2} * dC^{(3)})\,, \tag{2.23}$$

where each Hodge star appearing on the right hand side is with respect to the *un-warped* spacetime and internal metrics.

[1]Note that $H^{3,0}(\mathcal{M})$ is empty for a simply connected Calabi-Yau.

Combining (2.23) with the right hand side of (2.21) gives the following set of equations:

$$\begin{aligned}
\psi\, G \wedge dC^{(3)} &= 0, \\
d\psi \wedge G \wedge C^{(3)} &= 0, \\
d\psi \wedge d * C^{(3)} &= 0, \\
d * d\psi \wedge *C^{(3)} &= *\psi \wedge d\left(e^{-3\phi/2}\{*dC^{(3)} - dC^{(3)}\}\right).
\end{aligned} \tag{2.24}$$

To satisfy the third equation in (2.24), we can fix the gauge by demanding that,

$$d * \delta C = 0\,. \tag{2.25}$$

This choice differs from the usual gauge fixing condition (2.18) by an exact form. As a consequence of (2.25), $d * C^{(3)} = 0$. The first equation in (2.24) is a consequence of equation two which requires that,

$$G \wedge C^{(3)} = 0\,. \tag{2.26}$$

This condition is the analogue of the primitivity condition (2.10) for the metric.

The right hand side of the final equation in (2.24) must vanish since this term gives a mass to the spacetime field ψ. Requiring that the perturbation $dC^{(3)}$ take us to a supersymmetric vacuum implies that $dC^{(3)}$ must be self-dual, which in turn implies that $C^{(3)}$ is harmonic. Therefore any element of $H^{2,1}(\mathcal{M})$ which satisfies (2.26) gives rise to a chiral multiplet.

2.4 Lifting G-flux to F theory

If the eight manifold $\mathcal{M}$ is elliptically-fibered with base $\mathcal{B}$, then by shrinking the volume of the fiber, we can lift our M theory compactification to a four-dimensional type IIB compactification on $\mathbb{R}^4 \times \mathcal{B}$. Since the power of the warp factor is different for the spacetime and internal metric in (2.16), we might worry that the resulting four-dimensional metric breaks Lorentz invariance.

Let us first show that this is not the case. M theory on T^2 with area A maps to type IIB on a circle S^1 with radius proportional to $1/A$. Locally, the warp factor rescales the metric sending:

$$A \rightarrow e^{\phi/2} A\,.$$

However, this corresponds to rescaling the IIB circle metric by

$$\frac{1}{A^2} \rightarrow e^{-\phi}\frac{1}{A^2}\,,$$

which is precisely the power needed to obtain a Lorentz invariant four-dimensional metric.

How does the G-flux lift to type IIB? Let us start with the $G_{\mu\nu\rho a}$ component. This component has the form $dC_{\mu\nu\rho}$, and the this C-field lifts to a component of the four-form D^+ of type IIB:

$$C_{\mu\nu\rho} \to D^+_{\mu\nu\rho\lambda} \,. \tag{2.27}$$

Note that λ is a spacetime index and that dD^+ is not required to be self-dual. What is required to be self-dual is the combination,

$$F^+ = dD^+ - \frac{1}{2} B' \wedge H + \frac{1}{2} B \wedge H' \,. \tag{2.28}$$

Therefore the presence of this spacetime D^+ field does not imply that there is a D^+ field in the internal space.

We can divide the remaining G-flux involving $G_{a\bar{b}c\bar{d}}$ into two cases. There could be a component of G with no legs along the fiber. This component would map in the following way,

$$G_{a\bar{b}c\bar{d}} \to (dD)^+_{a\bar{b}c\bar{d}\lambda}.$$

This flux breaks four-dimensional Lorentz invariance. By self-duality of G, this case also rules out the possibility of components with two legs along the fiber.

The remaining possibility is the case where G is locally the product of a three-form on $\mathcal{B}$ and a one-form in the fiber. In this case, we can also differentiate between two kinds of G-flux. We can see this already in the relation between M theory on $K3 = T^4/\mathbb{Z}_2$ and the type IIB orientifold of T^2 by $\Omega(-1)^{F_L}\mathbb{Z}_2$, where Ω is world-sheet parity [19]. M theory on $T^4/\mathbb{Z}_2$ has 22 gauge-fields obtained by reducing the C-field on the 22 forms in $H^2(K3,\mathbb{Z})$. The 22 forms on $T^4/\mathbb{Z}_2$ are grouped in the following way: there are 16 twisted sector $(1,1)$ forms. Each twisted sector form comes from an A_1 singularity so the gauge group is enhanced to SU(2). The $(2,0)$ and $(0,2)$ forms descend from T^4 as do 4 more untwisted $(1,1)$ forms. The gauge group is $\mathrm{SU}(2)^{16} \times \mathrm{U}(1)^6$.

In the limit where $A \to 0$, the two untwisted sector forms corresponding to the class of the fiber and its Hodge dual are no longer normalizable. We can identify the remaining 20 forms with gauge-fields in the orientifold theory in the following manner: note that the action $\Omega(-1)^{F_L}$ is an element of $\mathrm{SL}(2,\mathbb{Z})$ given by the matrix,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{2.29}$$

which projects out both B and B'. The only components which survive the projection have a leg along T^2. The 4 Kaluza-Klein gauge-fields obtained by reducing B and B' along one-cycles of T^2 are then identified with the 4 surviving untwisted sector forms of $T^4/\mathbb{Z}_2$. The 16 fixed points coalesce into 4 groups of coincident A_1 singularities. The gauge group is enhanced from $\mathrm{SU}(2)^4 \to \mathrm{SO}(8)$. The SO(8) arises in the orientifold picture from placing 4 $D7$-branes at the location of each $O7$-plane.

Therefore, if our G-flux is localized around a singular fiber of the elliptic fibration, it will lift to the field strength of a seven-brane gauge-field. The gauge-field will have non-zero instanton number on the 4-cycle wrapped by the seven-brane. If there are multiple seven-branes then the gauge-group can be non-abelian as in the $T^4/\mathbb{Z}_2$ example. In this situation, the supergravity analysis is incomplete since the enhanced gauge symmetry is non-perturbative in M theory. From F theory, we certainly expect the gauge-field to satisfy the non-abelian Donaldson-Uhlenbeck-Yau equation. Any holomorphic stable vector bundle would then give a supersymmetric solution.

For the most part, we will consider the last choice for G. In this case, we can express G locally in terms of a basis for one-forms on the torus,

$$dz = dx + \tau dy\,, \qquad d\overline{z} = dx + \bar{\tau} dy\,,$$

in the following way,

$$\frac{G}{2\pi} = dz \wedge \omega - d\overline{z} \wedge *\omega\,, \tag{2.30}$$

where $\omega \in H^{1,2}(\mathcal{B})$. The flux then lifts to a combination of the NS field strength H and RR field strength H'. The field strengths are given in terms of ω,

$$H = \omega - *\omega \qquad H' = \omega\tau - *\omega\bar{\tau}. \tag{2.31}$$

The anomaly cancellation condition then becomes,

$$\frac{\chi}{24} = n - \int H \wedge H'\,, \tag{2.32}$$

where n is the number of D3-branes.

The NS field strength H and the RR field strength H' are naturally arranged in an $\mathrm{SL}(2,\mathbb{Z})$ doublet of type IIB supergravity:[2]

$$\begin{aligned} \Lambda &= \frac{1}{\sqrt{\tau_2}}\,(H' - \tau H)\,, \\ \Lambda^* &= \frac{1}{\sqrt{\tau_2}}\,(H' - \bar{\tau} H)\,. \end{aligned} \tag{2.33}$$

In terms of ω,

$$\Lambda \sim \sqrt{\tau_2}\,(*\omega)\,, \qquad \Lambda^* \sim \sqrt{\tau_2}\,\omega\,.$$

In a generic F theory compactification, the IIB fields undergo non-trivial monodromies by elements of $\mathrm{SL}(2,\mathbb{Z})$ around singular fibers. Under an $\mathrm{SL}(2,\mathbb{Z})$ transformation

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}\,,$$

[2]Unfortunately, the natural complex combinations of H and H' are usually denoted G and G^* in the literature. To avoid confusion, we have chosen the notation Λ and Λ^*. See [20] for an explanation of the mapping between the $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ and the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ parametrizations of the supergravity moduli space.

Λ and Λ^* transform in the following way,

$$\Lambda \to \Lambda \left(\frac{c\bar{\tau}+d}{c\tau+d}\right)^{1/2}, \qquad \Lambda^* \to \Lambda^* \left(\frac{c\bar{\tau}+d}{c\tau+d}\right)^{-1/2}. \tag{2.34}$$

Our H and H' field strengths can therefore be rotated non-trivially by $\mathrm{SL}(2,\mathbb{Z})$ as we move along the base space $\mathcal{B}$.

3. Constructing vacua with G-flux

3.1 $K3 \times K3$

The first example that we will consider is M theory on $K3_1 \times K3_2$. This compactification space gives $N = 4$ supersymmetry in three dimensions. The anomaly $\chi/24 = 24$ which must be cancelled by a combination of branes and G-flux. We therefore require that,

$$\frac{1}{2}\int \frac{G}{2\pi} \wedge \frac{G}{2\pi} \leq 24\,.$$

Let J_1 and J_2 denote the Kähler forms for $K3_1$ and $K3_2$, respectively. We will initially choose G-flux of the form,

$$\frac{G}{2\pi} = \omega_1 \wedge \omega_2\,, \tag{3.1}$$

where $\omega_i \in H^{1,1}(K3_i, \mathbb{Z})$. Each ω_i must also be primitive with respect to J_i. By definition, each ω_i is an element of the Picard group $\mathrm{Pic}(K3_i)$.

For a generic $K3$, the Picard group will be empty. To find a solution, let us construct a $K3$ surface in the following way: take a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2,2,2)$. There are three natural elements of $\mathrm{Pic}(K3)$ which we will denote C_1, C_2, C_3. These classes are determined by the three hyperplanes,

$$\begin{aligned} &\{p\} \times \mathbb{P}^1 \times \mathbb{P}^1\,,\\ &\mathbb{P}^1 \times \{p\} \times \mathbb{P}^1\,,\\ &\mathbb{P}^1 \times \mathbb{P}^1 \times \{p\}\,, \end{aligned} \tag{3.2}$$

for $\{p\}$ a point. With a standard abuse of notation, we will use C_i to denote both the cohomology class and the cycle dual to the Poincaré dual of C_i. The intersection matrix for the C_i is easily computed. Any two distinct C_i intersect on a $\mathbb{P}^1$. A quadratic in $\mathbb{P}^1$ gives two points. The self-intersection number of any C_i vanishes, so we obtain the following intersection matrix:

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}. \tag{3.3}$$

We can take $J = C_1 + C_2 + C_3$ as the Kähler form for this surface. As our basic primitive class, let us take $\alpha = C_1 - C_2$. The self-intersection number of α is -4.

We can take both $K3_1$ and $K3_2$ to be surfaces of the kind described above. Each space is endowed with a primitive form denoted α_1 and α_2. To cancel the anomaly completely, we can place 16 membranes on $K3_1 \times K3_2$ and turn on,

$$\frac{G}{2\pi} = \alpha_1 \wedge \alpha_2 \,. \tag{3.4}$$

We can also cancel the anomaly completely without branes in the following way: in addition to α, let us consider the primitive class $\beta = C_1 - C_3$ with self-intersection -4. Then $\alpha \cdot \beta = -2$ and we can turn on the flux,

$$\frac{G}{2\pi} = (\alpha_1 + \beta_1) \wedge \alpha_2 \,. \tag{3.5}$$

Note that this choice of G-flux is a primitive $(2,2)$ class with respect to each of the $\mathbb{P}^1 \times \mathbb{P}^1$ choices of complex structure. The full $N = 4$ supersymmetry is therefore preserved.

As a second more exotic example, let us consider the $K3$ surface obtained by quotienting a square T^4 with coordinates (z^1, z^2) by,

$$g_1 : \, (z^1, z^2) \to (iz^1, -iz^2) \,. \tag{3.6}$$

Under this $\mathbb{Z}_4$ quotient action, there are no untwisted $(1,1)$ forms. The resulting $K3$ has Picard number 20 [21]. Linear combinations of the twenty twisted sector $(1,1)$ forms are therefore integral classes. This implies that combinations of $(2,0)$ and $(0,2)$ forms are also integral classes because $H^2(K3,\mathbb{Z})$ has 22 elements. The intersection matrix for these transcendental integral classes is given by [21]

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} .$$

For this orbifold case, we note that the untwisted sector holomorphic $(2,0)$ form

$$\gamma = dz^1 dz^2 \tag{3.7}$$

satisfies $\int \gamma \wedge \bar{\gamma} = 4$. Let us take both $K3_1$ and $K3_2$ to be $T^4/\mathbb{Z}_4$ orbifolds. Then the $(2,2)$ form

$$\{\gamma_1 \wedge \bar{\gamma}_2 + \bar{\gamma}_1 \wedge \gamma_2\} \,, \tag{3.8}$$

defined on $K3_1 \times K3_2$ is primitive and integral.

We also require a class λ, which we take to be a primitive $(1,1)$ class with self-intersection -4. For example, the class obtained by taking the difference of the cycles coming from the resolution of the

$$\left(z^1 = \frac{1}{2}, \, z^2 = \frac{1}{2}\right), \qquad \left(z^1 = \frac{i}{2}, \, z^2 = \frac{i}{2}\right)$$

fixed points in the $(g_1)^2$ twisted sector. This class has zero intersection with every other $(1,1)$ form. There are a number of other choices for λ. We can then cancel the anomaly completely with the following G-flux:

$$\frac{G}{2\pi} = \lambda_1 \wedge \lambda_2 + \gamma_1 \wedge \bar{\gamma}_2 + \bar{\gamma}_1 \wedge \gamma_2 \,. \tag{3.9}$$

Note that this choice of G-flux does not preserve the full $N = 4$ supersymmetry. Varying the complex structure of $K3_i$ rotates $\gamma_i, \bar{\gamma}_i$ and J_i into each other. The resulting G is no longer supersymmetric. Therefore, only $N = 2$ supersymmetry survives. We will contruct another example of a $K3 \times K3$ compactification with G-flux in the following section.

Before leaving this case, let us see how special choices of G-flux appear in the $E_8 \times E_8$ heterotic dual. For illustration, let us take the form (3.1) for our G-flux. M theory on $K3_1 \times K3_2$ has a dual realization in terms of the heterotic string on $T^3 \times K3_2$. Away from points of enhanced symmetry, the heterotic string on T^3 has 22 abelian gauge-fields. As mentioned before, the gauge-fields arise in M theory by reducing the C-field on the 22 elements of $H^2(K3, \mathbb{Z})$. We can then express G reduced on $K3_1$ in terms of the field strength F for an abelian gauge-field,

$$\frac{G}{2\pi} = F \wedge \omega_1 \,.$$

We then take $F = \omega_2$, which corresponds on the heterotic side to taking an abelian connection with some instanton number on $K3_2$. Any membranes used to cancel the anomaly correspond to heterotic five-branes wrapping T^3. For the 16 gauge-fields coming from the Cartan of $E_8 \times E_8$, supersymmetry requires that,

$$g^{a\bar{b}} F_{a\bar{b}} = 0 \,,$$

and that F be a $(1,1)$ form. Clearly these constraints are satisfied by any G-flux of the form (3.1).

3.2 Some orbifold examples with constant G-flux

The next class of examples that we will construct have orbifold singularities. These examples all have *constant* G-fluxes. In cases with F theory lifts, the corresponding H and H'-fluxes will be also be constant. Let (z^1, z^2, z^3, z^4) coordinatize T^8. Since we will consider only $\mathbb{Z}_2$ quotients, we restrict T^8 to $T^2 \times T^2 \times T^2 \times T^2$ with each T^2 rectangular. We choose each T^2 to have periods,

$$\int_{\gamma_x^j} dz^i = \delta^i_j \,, \qquad \int_{\gamma_y^j} dz^i = i\, \delta^i_j \,,$$

where γ_x^i and γ_y^i are the x and y one-cycles.

Since our spaces will be $\mathbb{Z}_2$ quotients of T^8, the metric is flat away from the singularities and the Kähler form is,

$$J = \sum_i dz^i \wedge d\bar{z}^i \,. \tag{3.10}$$

Let us take G to have the form,

$$\begin{aligned} \frac{G}{2\pi} &= A\, d\bar{z}^1 dz^2 d\bar{z}^3 dz^4 + A^*\, dz^1 d\bar{z}^2 dz^3 d\bar{z}^4 + B\, d\bar{z}^1 dz^2 dz^3 d\bar{z}^4 + \\ &\quad + B^*\, dz^1 d\bar{z}^2 d\bar{z}^3 dz^4 + C\, d\bar{z}^1 d\bar{z}^2 dz^3 dz^4 + C^*\, dz^1 dz^2 d\bar{z}^3 d\bar{z}^4 \,. \end{aligned} \tag{3.11}$$

This choice of $(2,2)$ form certainly satisfies $J \wedge G = 0$. By construction, G is real. We also require that $G/2\pi$ be (half)-integer quantized. Requiring that $G/2\pi$ be integral over all four-cycles of T^8 gives the conditions:

$$2\left\{\mathrm{Re}A \pm \mathrm{Re}B \pm \mathrm{Re}C\right\} \in \mathbb{Z}\,, \qquad 2\left\{\mathrm{Im}A \pm \mathrm{Im}B \pm \mathrm{Im}C\right\} \in \mathbb{Z}\,. \tag{3.12}$$

The anomaly condition becomes,

$$16\left\{|A|^2 + |B|^2 + |C|^2\right\} + n = \frac{\chi}{24}\,, \tag{3.13}$$

where n is the number of branes.

Since we will consider orbifolds of T^8, we also need to ensure that $G/2\pi$ has (half)-integer intersections with all 4-cycles coming from the twisted sectors. We will need to check this condition in a case by case basis, but one possibility can be removed immediately. Certain twisted sectors can give rise to operators $\mathcal{O}$ which correspond to $(2,2)$ forms. However the two-point function of $\mathcal{O}$ with G satisfies,

$$\langle G\, \mathcal{O} \rangle = 0\,, \tag{3.14}$$

because $\mathcal{O}$ is charged under the orbifold gauge group. The remaining possibility is three-point functions of the form,

$$\langle G\, \mathcal{P}\, \mathcal{P}' \rangle\,, \tag{3.15}$$

where the two-forms $\mathcal{P}$ and $\mathcal{P}'$ carry opposite discrete charge.

As a first example of this kind, let us revisit $K3_1 \times K3_2$ where we realize $K3_i$ by $T^4/\mathbb{Z}_2$. We therefore quotient T^8 by $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by,

$$\begin{aligned} g_1 &: \quad (z^1, z^2) \to (-z^1, -z^2)\,, \\ g_2 &: \quad (z^3, z^4) \to (-z^3, -z^4)\,. \end{aligned} \tag{3.16}$$

The form (3.11) is invariant under this action, and for appropriate choices of A, B and C, we obtain supersymmetric compactifications. For example, the choice

$$A = 1 + \frac{i}{2}\,, \qquad B = \frac{1}{2}\,, \qquad C = 0\,,$$

JHEP08(1999)023

cancels the anomaly without any branes. To check that $G/2\pi$ is an integer form, we need to compute (3.15). First we note that $G \wedge \omega_1 = G \wedge \omega_2 = 0$ for ω_i the volume form of $K3_i$. This guarantees that if $\mathcal{P}$ and $\mathcal{P}'$ are two-forms from the same $K3$, (3.15) vanishes. The remaining possibility is when $\mathcal{P}$ and $\mathcal{P}'$ are charged under different $\mathbb{Z}_2$ actions in which case (3.15) vanishes by charge conservation.

A more interesting example is the $\mathbb{Z}_2$ quotient by the action,

$$g_1 : \; (z^1, z^2, z^3, z^4) \to (-z^1, -z^2, -z^3, -z^4) \,.$$

This space has singularities which cannot be resolved. However, it is perfectly fine as an M theory or type IIA compactification. String orbifold techniques give $\chi/24 = 16$. The Hodge numbers are:

$$H^{2,0} = 6\,, \qquad H^{1,1} = 16\,, \qquad H^{2,1} = 0\,, \qquad H^{3,1} = 16\,, \qquad H^{2,2} = 292\,.$$

As usual, $H^{4,0} = 1$. Tuning A, B and C appropriately gives solutions that cancel the tadpole either partially or completely. For example, the choice $A = 1$ completely cancels the tadpole with just G-flux. Checking that $G/2\pi$ is integer is easy in this case because there are no twisted sector two-forms $\mathcal{P}$. The only operators are four-forms $\mathcal{O}$ whose intersection with $G/2\pi$ vanishes.

Our next example is the symmetric product of $K3$. We therefore quotient by the action,

$$\begin{aligned} g_1 :&\;\; (z^1, z^2) \to (-z^1, -z^2) \\ g_2 :&\;\; (z^3, z^4) \to (-z^3, -z^4) \\ g_3 :&\;\; (z^1, z^2, z^3, z^4) \to (z^3, z^4, z^1, z^2)\,. \end{aligned} \tag{3.17}$$

This compactification space has $N = 3$ supersymmetry in three dimensions because $S^2(K3)$ is a hyperKähler space. In this case, $\chi/24 = 27/2$. The Hodge numbers are:

$$H^{2,0} = 1\,, \qquad H^{1,1} = 21\,, \qquad H^{2,1} = 0\,, \qquad H^{3,1} = 21\,, \qquad H^{2,2} = 232\,.$$

To obtain a consistent compactification, we therefore need to turn on G-flux. Note that invariance under the g_3 action requires that B and C be real. In this case, $G/2\pi$ can be half-integer quantized [16]. At first sight it seems that this additional freedom is not enough to find a solution to (3.13). However, the integrality condition

$$4\,\{\mathrm{Re}A \pm \mathrm{Re}B \pm \mathrm{Re}C\} \in \mathbb{Z}\,, \qquad 4\,\{\mathrm{Im}A\} \in \mathbb{Z} \tag{3.18}$$

permits the possibility,

$$A = \frac{1}{4}\,, \qquad B = \frac{5}{8}\,, \qquad C = \frac{5}{8}\,. \tag{3.19}$$

With this choice $G/2\pi$ survives the quotienting and can cancel the anomaly completely. An alternative is to set $C = 0$ and add a single membrane.

To confirm that $G/2\pi$ is indeed half-integral, we need to check that the intersection of $G/2\pi$ with all twisted sector states is also half-integral. The intersections with the twisted sector states corresponding to the generators g_1 and g_2 vanish by exactly the arguments given for the $T^8/(\mathbb{Z}_2)^2$ example above. We therefore only need to check the condition for twisted sector states generated by g_3. For this purpose, it is easier to consider $S^2(T^4)$ rather than $S^2(K3)$. This simplification is possible because the unique twisted sector $(1,1)$ form on $S^2(K3)$ descends from the unique twisted sector $(1,1)$ form on $S^2(T^4)$.

Denote $S^2(T^4)$ by X and let $\widetilde{X}$ be its resolution obtained by blowing up over the fixed locus which is the diagonal four-torus T^4_D. Let[3] $Y = T^4 \times T^4$ and $\widetilde{Y}$ be the space obtained by blowing up Y over the diagonal T^4_D. There are projections q and p from $\widetilde{Y}$ to Y and $\widetilde{X}$ to X respectively. The involution $i : Y \to X$ lifts to an involution $s : \widetilde{Y} \to \widetilde{X}$, which is branched over the exceptional divisor. We can summarize this information in the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{s} & \widetilde{X} \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{i} & X\,. \end{array} \tag{3.20}$$

Now consider a form $\omega \in H^{2,2}(Y) \cap H^4(Y,\mathbb{Z})$. For example, the form defined in (3.11). This form can be pulled back to $\widetilde{Y}$ giving $q^*\omega \in H^{2,2}(\widetilde{Y})$ and then pushed forward to a form:

$$\widetilde{\omega} = s_*(q^*\omega) \in H^{2,2}(\widetilde{X})\,.$$

Note that $\widetilde{Y}$ is a double cover of $\widetilde{X}$. Therefore $s^*(s_*v) = 2v$ for any form v on $\widetilde{Y}$ and we have $s^*\widetilde{\omega} = 2q^*\omega$. Consider the integral twisted sector $(1,1)$ form $\widetilde{t} \in H^{1,1}(\widetilde{X}) \cap H^4(\widetilde{X},\mathbb{Z})$ which is Poincaré dual to the exceptional divisor in $\widetilde{X}$. Then

$$s^*\widetilde{t} = 2t\,,$$

where $t \in H^{1,1}(\widetilde{Y}) \cap H^4(\widetilde{Y},\mathbb{Z})$ is Poincaré dual to the exceptional divisor in $\widetilde{Y}$. This is true again because $\widetilde{Y}$ is a double cover of $\widetilde{X}$ and s is branched over the exceptional divisor. The relevant three point function we wish to compute is

$$\langle \widetilde{t}^2 \cdot \widetilde{\omega}\rangle_{\widetilde{X}} = \frac{1}{2}\langle s^*(\widetilde{t}^2 \cdot \widetilde{\omega})\rangle_{\widetilde{Y}} = 2\langle t^2 \cdot s^*\widetilde{\omega}\rangle_{\widetilde{Y}} = 4\langle t^2 \cdot q^*\omega\rangle_{\widetilde{Y}}\,.$$

Since the normal bundle of the diagonal T^4_D is trivial, we see that the exceptional divisor D corresponding to t is just $\mathbb{P}^1 \times T^4_D$. The intersection is then

$$-8\langle T^4_D \cdot q^*\omega\rangle_{\widetilde{Y}} = -8\langle T^4_D \cdot \omega\rangle_Y\,.$$

[3]We wish to thank D.-E. Diaconescu for explaining the following argument.

But this can now be readily computed: first it is easy to check that the contributions from the A and A^* terms vanish because the forms themselves vanish on T^4_D. The other terms yield the following integrality condition for half-integral G-flux,

$$128(B + C) \in \mathbb{Z} \,.$$

While this was derived for $S^2(T^4)$, the same condition can be seen to hold for the symmetric product of $K3$ in the orbifold limit. This condition is satisfied for our choice (3.19) of G-flux.[4]

Lastly, we consider an example which gives $N = 2$ supersymmetry in three dimensions. We quotient by,

$$\begin{aligned} g_1 :&\ \ (z^1, z^2) \to (-z^1, -z^2)\,, \\ g_2 :&\ \ (z^1, z^3) \to (-z^1, -z^3)\,, \\ g_3 :&\ \ (z^2, z^4) \to (-z^2, -z^4)\,. \end{aligned} \tag{3.21}$$

The quotient group essentially inverts all possible pairs of tori. For this case, $\chi/24 = 28$. The Hodge numbers are:

$$H^{2,0} = 0\,, \qquad H^{1,1} = 100\,, \qquad H^{2,1} = 0\,, \qquad H^{3,1} = 4\,, \qquad H^{2,2} = 460\,.$$

As a sample choice of G-flux, we can take

$$A = 1 + \frac{i}{2}\,, \qquad B = \frac{1}{2} + \frac{i}{2} \qquad C = 0\,,$$

which cancels the anomaly completely. To check that $G/2\pi$ is integer, we need to check that it has integer periods over all integer homology cycles. For the untwisted sector cycles, this reduces to verifying (3.12), which is obvious. For the twisted sector cycles, it is easy to repeat the arguments presented for the previous $T^8/(\mathbb{Z}_2)^2$ example to show that all three point functions of type (3.15) vanish.

3.3 An orientifold example

In a similar way, we can construct four-dimensional orientifold examples with constant fluxes. We start with an orbifold of the form T^6/Γ. Again for simplicity let us take $T^6 = T^2 \times T^2 \times T^2$ with each factor rectangular and coordinates (z^1, z^2, z^3). We

[4]The symmetric product is a highly singular space. However, we can smooth the space by blowing up the symmetric product $X = S^2(K3)$ over the fixed locus. This amounts to replacing X by the Hilbert scheme of points $K3^{[2]}$. See, for example, [22]. In a diagram analogous to (3.20), $\widetilde{X} = K3^{[2]}$ and $p : K3^{[2]} \to X$. If we want $p^*(G/\pi)$ to be a primitive element of $H^{2,2}(K3^{[2]}, \mathbb{Z})$, we need to impose at least one additional condition on (3.11). The Kähler class of $K3^{[2]}$ has a term proportional to the class of the exceptional divisor. To ensure primitivity, we can require that G/π vanish on the fixed locus, which implies that $B = C$. This is a natural way to get a smooth hyperKähler compactification. We wish to thank L. Göttsche for pointing out this generalization.

JHEP08(1999)023

can view an orientifold of T^6/Γ as a special point in the moduli space of F theory compactified on the elliptically-fibered four-fold [6],

$$\mathcal{M} = \frac{T^6/\Gamma \times T^2}{\mathbb{Z}_2},$$

where the $\mathbb{Z}_2$ action inverts both z^3 and the coordinate, z^4, of the fiber T^2. This F theory compactification reduces to the orientifold of type IIB on T^6/Γ by the action $\Omega(-1)^{F_L}\mathbb{Z}_2$ where the $\mathbb{Z}_2$ inverts z^3. This action produces various $O7$-planes at complex codimension one fixed sets on T^6. By adding $D7$-planes, we can cancel the $O7$-plane charge. Fixed points under group elements in the product of $\Omega(-1)^{F_L}\mathbb{Z}_2$ and Γ generate $O3$-planes. The total $D3$-brane tadpole is $\chi(\mathcal{M})/24$.

As a specific example, we can take the following generators for Γ,

$$\begin{aligned} g_1 &: \ (z^1, z^2) \to (-z^1, -z^2) \\ g_2 &: \ (z^1, z^3) \to (-z^1, -z^3)\,. \end{aligned} \tag{3.22}$$

The resulting Calabi-Yau T^6/Γ has Hodge numbers $h^{1,1} = 51$, $h^{2,1} = 3$. The associated four-fold $\mathcal{M}$ is the final example of section 3.2. This orientifold and its relation to F theory has been studied in [23, 24]. To cancel the anomaly without using any $D3$-branes, we can turn on the fluxes

$$\begin{aligned} H &= A\, d\overline{z}^1 dz^2 d\overline{z}^3 + A^*\, dz^1 d\overline{z}^2 dz^3 + B\, d\overline{z}^1 dz^2 dz^3 + B^*\, dz^1 d\overline{z}^2 d\overline{z}^3\,, \\ H' &= Ai\, d\overline{z}^1 dz^2 d\overline{z}^3 - A^* i\, dz^1 d\overline{z}^2 dz^3 - Bi\, d\overline{z}^1 dz^2 dz^3 + B^* i\, dz^1 d\overline{z}^2 d\overline{z}^3\,, \end{aligned} \tag{3.23}$$

with the choice:

$$A = 1 + \frac{i}{2}, \qquad B = \frac{1}{2} + \frac{i}{2}.$$

It is interesting to note that the same four-fold can give rise to many different orientifolds depending on the choice of C and G-flux. This point will be explored more fully elsewhere.

4. A heterotic compactification with torsion

In this section, we will construct an example of a four-dimensional SO(32) heterotic string compactification with torsion. This particular example has either $N = 1$ or $N = 2$ spacetime supersymmetry, depending on the choice of flux. We begin with type IIB compactified on an orientifold of $K3 \times T^2$. After a series of T and S dualities, we will arrive at our heterotic model.

The initial IIB supergravity metric is conformal to the metric on $K3 \times T^2$. The solution still possesses two isometries along the T^2. Two T-dualities along the two circles of T^2 sends $\Omega(-1)^{F_L}\mathbb{Z}_2$ to Ω [19]. In other words, it takes our F theory compactification with fluxes to a type I theory. In the subsequent discussion, we will specify the resulting type I background. If we choose to use only $D3$-branes and no background flux to cancel the anomaly, the resulting theory is type I on $K3\times T^2$ with 24 $D5$-branes wrapping the T^2. With background fluxes, the result is quite different. We will find type I compactified on a space $\mathcal{B}'$ with the following properties:

1. It is a complex manifold which is not Kähler, or even conformally Calabi-Yau.

2. It has vanishing first Chern class.

3. It has a non-zero H'-flux.

After a further S-duality, we end up with the SO(32) heterotic string compactified on $\mathcal{B}'$ with a non-zero H-flux.

4.1 Mapping the parameters and couplings

Let us consider the orientifold of type IIB on $K3\times T^2$ with a square T^2 by the action $\Omega(-1)^{F_L}\mathbb{Z}_2$. The $\mathbb{Z}_2$ action sends $z^1 \to -z^1$ where z^1 is the coordinate of the T^2. This compactification is a special point in the moduli space of F theory on $K3\times K3$. Let the T^2 have sides of length R_1 and R_2, and volume $\widetilde{V} = R_1 R_2$. We will take the $K3$ to have volume V. At the orientifold point, the ten-dimensional string coupling is a free parameter which we will take to be g_B.

Two T-duality transformations along T^2 invert the radii in the usual way,

$$R_i \to \frac{\alpha'}{R_i},$$

where $i = 1, 2$. The resulting type I theory has the following couplings and volumes:

$$g_I^{(4)} = \frac{g_B}{\sqrt{V\widetilde{V}}}, \qquad g_I^{(10)} = \frac{\alpha' g_B}{\widetilde{V}}, \qquad \widetilde{V}_I = \frac{\alpha'^2}{\widetilde{V}}, \qquad V_I = V. \tag{4.1}$$

Here $g^{(4)}$ and $g^{(10)}$ denote the four and ten-dimensional couplings. The $O7$-planes and $D7$-branes are mapped to an $O9-D9$ system. If, for simplicity, we assume a trivial seven-brane gauge-field configuration over $K3_1$ then the initial gauge group is SO(8)^4. The gauge-fields of the resulting $O9-D9$ theory then have non-trivial Wilson lines. Under a further S-duality transformation, we get the heterotic SO(32) theory. The resulting couplings and volumes can again be written in terms of IIB variables,

$$g_{het}^{(4)} = \sqrt{\frac{g_B}{(V\alpha')}}, \qquad g_{het}^{(10)} = \frac{\widetilde{V}}{g_B\alpha'}, \qquad \widetilde{V}_{het} = \frac{\alpha'}{g_B}, \qquad V_{het} = \frac{V\widetilde{V}^2}{g_B^2\alpha'^2}. \tag{4.2}$$

Our initial IIB supergravity description is valid in the limit where,

$$\frac{\widetilde{V}}{\alpha'}, \ \frac{V}{(\alpha')^2} \gg 1. \tag{4.3}$$

If we want a weakly coupled orientifold theory, we can also take g_B to be small but that is not necessary. Under condition (4.3), both the heterotic and type I four-dimensional couplings can be made small. If g_B is small, then an α' expansion of the resulting heterotic string theory is a good approximation.

4.2 The type IIB solution

To obtain the type IIB supergravity metric, we begin with M theory on $\mathcal{M} = K3_1 \times T^4/\mathbb{Z}_2$. If we choose a smooth $K3_1$ then our resulting type IIB metric will be smooth. We can also consider orbifold cases where $K3_1 = T^4/\Gamma$ where the metric is explicitly known. Otherwise, our resulting heterotic solution is given in terms of the metric of $K3_1$. We will use coordinates w^a for $K3_1$. We again take $T^4 = T^2 \times T^2$ with coordinates (z^1, z^2) and each factor square. Our initial M theory metric is then of the form (2.16) with g the metric on $\mathcal{M}$. We can choose to completely or partially cancel the anomaly with G-flux satisfying,

$$\frac{1}{2}\int_{\mathcal{M}} \frac{G}{2\pi} \wedge \frac{G}{2\pi} + n = 24\,.$$

Using arguments along the lines discussed in section 3, we can construct a G with the form

$$\frac{G}{2\pi} = \alpha \wedge dz^1 d\overline{z}^2 + \alpha^* \wedge d\overline{z}^1 dz^2 + \beta \wedge d\overline{z}^1 d\overline{z}^2 + \beta^* \wedge dz^1 dz^2\,, \tag{4.4}$$

where $\alpha \in H^{1,1}(K3_1)$ and $\beta \in H^{2,0}(K3_1)$. Note that if $\beta = 0$, the model has $N = 2$ supersymmetry.

We treat the z^2 direction as the elliptic fiber, and lift this M theory vacuum to a type IIB orientifold of $K3_1 \times T^2$. The resulting background fluxes are given by,

$$\begin{aligned} H &= (\alpha + \beta^*) \wedge dz^1 + (\alpha^* + \beta) \wedge d\overline{z}^1\,, \\ H' &= (\bar\tau\alpha + \tau\beta^*) \wedge dz^1 + (\tau\alpha^* + \bar\tau\beta) \wedge d\overline{z}^1\,. \end{aligned} \tag{4.5}$$

We note that H and H' can be expressed in the form,

$$\begin{aligned} H &= d\left\{\Lambda_{\alpha+\beta^*} \wedge dz^1 + \Lambda_{\alpha^*+\beta} \wedge d\overline{z}^1\right\}, \\ H' &= d\left\{\Lambda_{\bar\tau\alpha+\tau\beta^*} \wedge dz^1 + \Lambda_{\tau\alpha^*+\bar\tau\beta} \wedge d\overline{z}^1\right\}. \end{aligned} \tag{4.6}$$

The potentials Λ_γ are not globally defined forms on the space $K3_1 \times T^2$, but satisfy $d\Lambda_\gamma = \gamma$.

In string frame, the type IIB supergravity metric has the form:

$$\begin{pmatrix} \Delta'\,\eta_{\mu\nu} & 0 \\ 0 & \Delta\,\widetilde{g} \end{pmatrix}. \tag{4.7}$$

The indices $\mu, \nu = 0, \ldots, 4$ and $\widetilde{g}$ is the metric of $K3_1 \times T^2/\mathbb{Z}_2$. The warp factors Δ and Δ' depend on the internal coordinates. We can determine Δ and Δ' in the following way. Let us reduce from M theory to type IIA along a side of the elliptic fiber. The metric of the torus is warped,

$$e^{\phi/2} dz^2 d\overline{z}^2 \, .$$

The resulting type IIA metric in string frame is given by [25],

$$g_{\rm IIA} = e^{\phi/4} G^{(10)} \, , \tag{4.8}$$

where $G^{(10)}$ is the straight dimensional reduction of the M theory metric. Using the metric (2.16), we find that:

$$\Delta = (\Delta')^{-1} = e^{3\phi/4} \, . \tag{4.9}$$

Lastly, let us recall that the warp factor is determined by equation (2.5). There are three source terms on the right hand side of this equation. Both the X_8 curvature term and the membrane term are suppressed by six powers of M_{pl} relative to the $G \wedge G$ source term. To leading order in the derivative expansion, we can neglect the effect of both terms.[5] With the form of G-flux given in (4.4), the warp factor will have no dependence on (z^1, z^2) at the level of the supergravity solution.

We can see this directly in type IIB supergravity. The only non-vanishing component of D^+ is given by,

$$D^+_{\mu\nu\rho\lambda} = \epsilon_{\mu\nu\rho\lambda} e^{-3\phi/2} .$$

The self-dual field strength F^+ given in (2.28) obeys [26],

$$\begin{aligned} d * F^+ = H \wedge H' + (4\pi^2\alpha')^2 \Big\{ \frac{1}{64\pi^2} \sum_{i=1}^{4} \mathrm{tr}(R \wedge R) \, \delta^2(z^1 - z^1_i) + \\ + \sum_{j=1}^{n} \delta^2(z^1 - z^1_j) \delta^4(w - w_j) \Big\} \, , \end{aligned} \tag{4.10}$$

where z^1_i are the locations of the $O7$-plane plus four $D7$-branes, and (z^1_j, w_j) are the locations of the $D3$-branes. From (4.10), we obtain an equation for the warp factor:

$$\begin{aligned} d * dD^+ = H \wedge H' + (4\pi^2\alpha')^2 \Big\{ \frac{1}{64\pi^2} \sum_{i=1}^{4} \mathrm{tr}(R \wedge R) \, \delta^2(z^1 - z^1_i) + \\ + \sum_{j=1}^{n} \delta^2(z^1 - z^1_j) \delta^4(w - w_j) \Big\} \, . \end{aligned} \tag{4.11}$$

Again the $H \wedge H'$ term is constant in z^1 while the remaining source terms are suppressed by powers of α'.

[5]It is worth noting that there is an obstruction to solving the warp factor equation (2.5). However, if the anomaly cancellation condition (1.2) is satisfied then the obstruction vanishes. The X_8 and membrane terms are then crucial for ensuring the existence of a solution for the warp factor.

JHEP08(1999)023

4.3 Dualizing to a type I solution

To arrive at a type I solution, we T-dualize along both sides of the T^2 with coordinate $z^1 = x^1 + iy^1$. Using Buscher's duality [27] and [28, 29, 30], we can express the type I metric g^I and RR two-form B'^I in terms of our initial type IIB quantities. Since H and H' can be expressed in the form (4.6), we only have B_{xa}, B'_{xa} and B_{ya}, B'_{ya} components. The type I metric is then given by,[6]

$$
\begin{aligned}
g^I_{ab} &= \Delta\tilde{g}_{ab} + \frac{1}{\Delta\tilde{g}_{xx}} B_{xa}B_{xb} + \frac{1}{\Delta\tilde{g}_{yy}} B_{ya}B_{yb}\,, \\
g^I_{xa} &= \frac{1}{\Delta\tilde{g}_{xx}} B_{xa}\,, \qquad\qquad g^I_{ya} = \frac{1}{\Delta\tilde{g}_{yy}} B_{ya}\,, \\
g^I_{xx} &= \frac{1}{\Delta\tilde{g}_{xx}}\,, \qquad\qquad g^I_{yy} = \frac{1}{\Delta\tilde{g}_{yy}}\,, \qquad g^I_{xy} = 0\,,
\end{aligned}
\tag{4.12}
$$

where the a, b directions are along $K3_1$. The type I dilaton is inversely proportional to the warp factor:

$$
e^{\phi^I} = \frac{g_B}{\Delta\sqrt{\tilde{g}_{xx}\tilde{g}_{yy}}}\,. \tag{4.13}
$$

The B'^I-field is given by,

$$
\begin{aligned}
B'^I_{ab} &= \frac{3}{2}\left\{B_{x[a}B'_{by]} - B'_{x[a}B_{by]}\right\} + 2B'_{x[a}B_{b]y}\,, \\
B'^I_{xa} &= B'_{ay}\,, \qquad B'^I_{ya} = -B'_{ax}\,, \qquad B'^I_{xy} = 0\,.
\end{aligned}
\tag{4.14}
$$

Note that we have used the fact that D^+ has no internal components in the above expressions. If we set either α or β to zero in (4.5), then one can check using the local expressions (4.6) for B and B' that $B'^I_{ab} = 0$.

Using coordinates w^a for $K3_1$, we can rewrite the type I metric in a way that makes the structure a little clearer:

$$
ds^2 = \Delta\tilde{g}_{ab}dw^a dw^b + \frac{1}{\Delta\tilde{g}_{xx}}\left(dx + B_{xa}dw^a\right)^2 + \frac{1}{\Delta\tilde{g}_{yy}}\left(dy + B_{ya}dw^a\right)^2. \tag{4.15}
$$

The T^2 parametrized by x and y is now non-trivially twisted over the base $K3_1$. By viewing $B_{xa}dw^a$ and $B_{ya}dw^a$ as Kaluza-Klein gauge-fields, we see that the twisting is encoded in the characteristic classes of these gauge-fields on $K3_1$.

Since T-duality is a perturbative symmetry, we have arrived at a consistent type I background (4.12), (4.13) and (4.14). We can S-dualize to a perturbative heterotic compactification with dilaton,

$$
e^{\phi_{het}} = \frac{1}{g_B}\Delta\sqrt{\tilde{g}_{xx}\tilde{g}_{yy}}\,, \tag{4.16}
$$

[6]We set $4\pi^2\alpha' = 1$ for the remainder of the paper.

and string frame metric:

$$\frac{1}{g_B}\sqrt{\widetilde{g}_{xx}\widetilde{g}_{yy}}\begin{pmatrix}\eta_{\mu\nu} & 0\\ 0 & \Delta\, g^I\end{pmatrix}. \tag{4.17}$$

Under S-duality, $B'^I \to B^{het}$.

Torsional string compactifications satisfy a number of stringent constraints. See [31, 32, 33, 34, 35] for a discussion of various aspects of torsional compactifications. As a final comment, we note that since our torsional vacuum is T-dual to an anomaly free orientifold, we can infer a great deal about the resulting metric and B-field. Our space should have trivial first Chern class. Since no SO(32) gauge-fields are excited, $p_1(\mathcal{B}')$ must also be trivial in cohomology to satisfy the usual anomaly cancellation condition,

$$dH^{het} = \mathrm{tr}(R\wedge R) - \frac{1}{30}\mathrm{Tr}(F\wedge F)\,.$$

This is not as implausible as it might first seem. In the $E_8\times E_8$ heterotic dual to the original type IIB orientifold, $p_1(K3)$ is cancelled by embedding instantons in some abelian gauge-fields. It seems reasonable that the instantons in the Kaluza-Klein gauge-fields $B_{xa}dw^a$ and $B_{ya}dw^a$ could analogously cancel $p_1(K3)$ in this case. Finally, it should be possible to show that the metric (4.17) and H^{het} are derivable from a real prepotential, analogous to the Kähler potential [31].

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